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Modified k-ϵ Model for Compressible Free Shear Flows

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Introduction

IN the landmark conference on turbulent free shear flows, held at the NASA Langley Research Center in 1972,[1] the k-ϵ model emerged as one of the most capable of the turbulence models available at the time.[2] However, one deficiency of the k-ϵ model was also brought to light. Namely, the inability to predict the experimentally observed[3] decrease in the spreading rate of turbulent, supersonic, free shear layers as the Mach number was increased. A modification to the k-ϵ model is proposed here that helps remove this deficiency.

Theory

Modifications to the k-ϵ model to account for compressibility effects have been proposed in the past by Dash et al.[4] and Chuech et al.,[5] among others. Sarkar et al.[6] and Zeman[7] have recently proposed dissipation models that account for compressibility effects. The adaptation of these ideas to the k-ϵ model is also straightforward. An alternative approach is presented here, based on a recent suggestion by Kim.[8]

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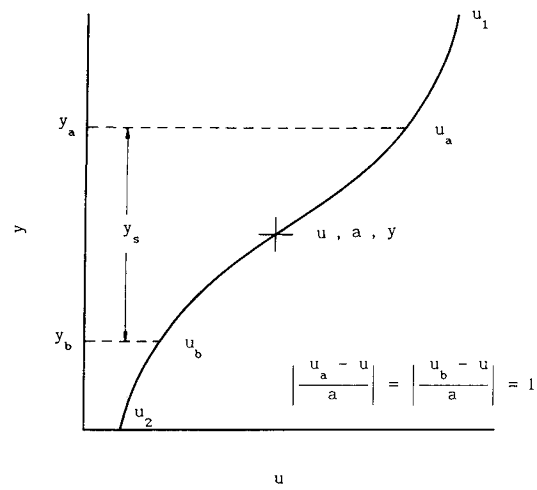

Fig. 1 Characteristic width for supersonic shear layers.[8]

Kim proposed a simple modification to another turbulence model, the Prandtl mixing length hypothesis (MLH). The MLH involves evaluating the turbulent viscosity by using a "mixing length" ℓ_m. This mixing length is usually computed by setting it equal to a constant fraction of some characteristic width y_G such that

$$\ell_m = \lambda y_G \tag{1}$$

where λ is a constant. For incompressible free shear layers, y_G is customarily taken as the width of the shear layer b, usually defined as

$$y_G = b + |y_{0.9} - y_{0.1}| \tag{2}$$

where at $y_{0.1}$, $u^* = 0.1$, and at $y_{0.9}$, $u^* = 0.9$, where $u^* = (u - u_2)/(u_1/ - u_2)$. Here u is the velocity, y is the transverse distance, and the subscripts 1 and 2 represent the high and low velocity sides of the shear layer, respectively.

Kim, however, proposed a modified formula and a different characteristic width y_s defined by

$$y_s = |y_a - y_b| \tag{3}$$

The locations of y_a and y_b were defined by points where sonic flow exists relative to the local point, as shown in Fig. 1. (In Fig. 1 a is the local speed of sound.) Limitations were placed on y_a and y_b in that these locations could not exceed the edges of the shear layer.

In the "standard" high Reynolds number k-ϵ model (designated the $k\epsilon1$ model in Ref. 2), the turbulent viscosity is computed from

$$\mu_t = C_\mu \rho \left(\frac{k^2}{\epsilon}\right) \tag{4}$$

where C_μ is a constant, ρ is the density, k is the turbulent kinetic energy, and ϵ is the turbulent kinetic energy dissipation rate. The values for k and ϵ are determined by solving a pair of partial differential transport equations. Comparing Eq. (4) with the Prandtl-Kolmogorov formula

$$\mu_t = \rho\sqrt{k}\,\ell \tag{5}$$

where ℓ is the length scale, shows that the length scale used in the $k\epsilon1$ model is determined from the relationship

$$\ell_\epsilon = C_\mu \frac{k^{1.5}}{\epsilon} \tag{6}$$

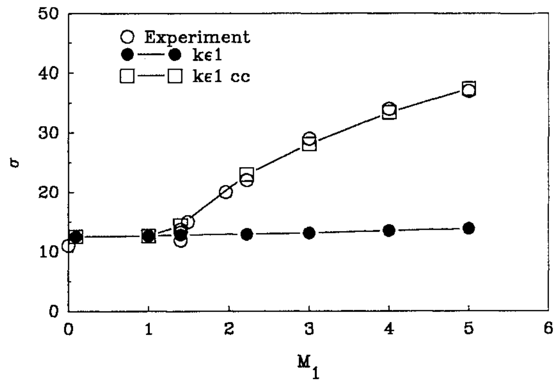

Fig. 2 Variation of spreading parameter σ with Mach number.

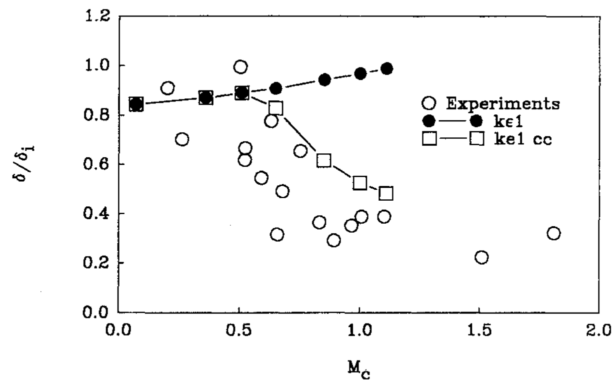

Fig. 3 Variation of normalized shear-layer growth rate with convective Mach number.

Now, a new length scale based on the characteristic width y_s can also be defined by

$$\ell_s = \lambda_s y_s \tag{7}$$

where λ_s is a new model constant. If no relative supersonic flow exists in the layer, λ_s is set equal to unity.

It is now proposed that the length scale for use in the modified k-ϵ model be taken as the smallest length scale determined from Eqs. (6) and (7). Should the local length scale from Eq. (7) be smaller than the usual value from Eq. (6), then the reduced length scale needs to be introduced into the k-ϵ model. From Eq. (6), this can be very easily accomplished by either reducing C_μ or increasing ϵ. The approach adopted here was to increase ϵ. Consequently, the dissipation rate in the modified model was computed from

$$\epsilon = \max\left[\epsilon_{\substack{\text{from transport}\\ \text{equation}}},\ C_\mu \frac{k^{1.5}}{\ell_s}\right] \tag{8}$$

It is interesting to note that under conditions where the length scale is effectively determined by Eq. (7), the original two-equation k-ϵ model is effectively reduced to a one-equation k model (see, for example, Ref. 2) where the length scale is now determined algebraically.

Results

The constant λ_s was selected by matching the calculated spreading-rate parameter of a Mach 5, single stream, air free shear layer with the experimentally observed value[3] of 37. This resulted in a value for λ_s of 0.0445.

To test the modified k-ϵ model (hereafter designated the $k\epsilon 1_{cc}$ model) the spreading rates of various single stream, air mixing layers (including a sample of those compiled for the Langley conference) were computed with the following assumptions: p = const, $T_{t1} = T_{t2}$, $Pr = Pr_t = 0.7$, $T_1 = 300$ K, and $u_2/u_1 = 0$, where p is the static pressure, T_t and T are the total and static temperatures, respectively, and Pr and Pr_t are the laminar and turbulent Prandtl numbers. The computed spreading parameter was determined from the following formula (as required of the "predictors" at the NASA Langley conference):

$$\sigma = 1.855 \frac{1}{\mathrm{d}b/\mathrm{d}x} \tag{9}$$

where b is defined by Eq. (2). The numerical computations were performed using a two-dimensional parabolic Navier-Stokes program described in Ref. 9. The standard high Reynolds number k-ϵ model constants[2] have been used.

Figure 2 shows the computed variation in σ with freestream Mach number M_1. The agreement between the $k\epsilon 1_{cc}$ model and the experimental data compiled for the Langley conference is excellent. The standard model, in comparison, displays virtually no variation in spreading parameter with Mach number.

Recently, there has been much success at correlating the spreading rates of two-stream supersonic shear layers by using the convective Mach number M_c as proposed by Bogdanoff[10] and Papamoschou and Roshko.[11] Figure 3 shows a selection of experimental normalized growth rates vs M_c as compiled by Samimy and Elliott.[12] The normalized growth rate is obtained by dividing the experimentally observed growth rate of the compressible shear layer (here defined as $\delta = \mathrm{d}b/\mathrm{d}x$) by the so-called "incompressible" growth rate δ_i, which would be observed in a subsonic shear layer with the same velocity and density ratios. The incompressible values are usually estimated by the formula[11]

$$\delta_i = \left(\frac{\mathrm{d}b}{\mathrm{d}x}\right)_i = \frac{C_\delta}{2} \frac{[1-(u_2/u_1)]\,(1+\sqrt{\rho_2/\rho_1})}{1+(u_2/u_1)\sqrt{\rho_2/\rho_1}} \tag{10}$$

where C_δ is a constant. To compare the computed results presented here with those obtained experimentally, the computed spreading rates have been similarly divided by the value of δ_i determined from Eq. (10). The presentation of the results in this way differs from the common practice of dividing the predicted compressible spreading rates by the predicted incompressible ones. It was felt that using Eq. (10) to evaluate δ_i provided a better comparison with the experimental data, which had been analyzed in a similar fashion. To normalize the computed spreading rates, a value of $C_\delta = 0.165$ was used in Eq. (10), so that $\delta_i = 0.165$ when $u_2/u_1 = 0$ and $\rho_2/\rho_1 = 1$ in agreement with the recommendations of Rodi.[13]

The numerical computations were performed with air in both streams and with the following assumptions: p = const, $T_{t1} = T_{t2}$, $Pr = Pr_t = 0.7$, $T_1 = 300$ K, $u_2/u_1 = 0.25$, and $Re_{u,1} = 4.4 \times 10^7\ \mathrm{m}^{-1}$ where Re_u is the unit Reynolds number. The results, also shown in Fig. 3, show that the $k\epsilon 1_{cc}$ model predicts the same trend as the experimental data. The significant improvement over predictions obtained with a standard $k\epsilon 1$ model are immediately obvious.

Conclusion

A modification to the two-equation $k\epsilon 1$ model was proposed that effectively reduced the model to a one-equation k model in highly compressible flows. The modification has been shown to improve the prediction ability of the $k\epsilon 1$ model when it is applied to supersonic shear layers.

Acknowledgments

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Taylor Series Approximation of Geometric Shape Variation for the Euler Equations

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Introduction

THE objective of the present study is the implementation and testing of a linear approximate analysis scheme for the Euler equations, where the treatment of variations in geometric shape is the focus. The method is potentially useful for engineers who must apply modern computational fluid dynamics (CFD) software in a design environment. Detailed documentation of the present work is found in Ref. 1. Approximate analysis is very closely related to a design-oriented discipline known as sensitivity analysis, which involves the calculation of slopes, known as sensitivity derivatives, which are defined as the derivatives of the responses of interest of a particular system with respect to the independent (design) variables of interest.[2-6]

Of particular interest and concern in this research is the use of slopes to predict solution changes in the presence of discontinuities in the flowfield (e.g., shock waves in supersonic flow). Although in the discretization of the governing equations these discontinuities are typically approximated by a continuously differentiable set of algebraic equations, nevertheless, the slopes which are found in a numerical solution in the neighborhood of shocks can fluctuate wildly. Thus a potential difficulty with the technique is identified at the outset, and its effect could vary depending on the particular flowfield and the strength of the discontinuities.

Presentation of Theory

The governing equations considered here are the two-dimensional Euler equations, and are solved in integral conservation law form using a cell-centered finite volume formulation.[7,8] Higher-order accurate upwind discretization of the spatial terms is accomplished using the continuously differentiable flux-vector splitting procedure of van Leer.[9] Numerical integration of the equations in time to steady state is implemented through implicit discretization and "delta" form linearization in time, resulting in the solution at each time step of a large system of linear equations. In essence, steady-state solution of the governing equations is replaced (approximately) by solution of a large system of simultaneous nonlinear algebraic equations over the domain, given as

$$\{R(Q^*)\} = \{0\} \tag{1}$$

where the vector $\{Q^*\}$, the "root," is the steady-state solution for the field variables.

In modern CFD solutions of fluid-flow problems, the geometric shape of the computational domain is defined by the computational mesh, which is generally "body-oriented." A computational mesh is of course defined by the complete set of (x,y) coordinates of the intersection points of the grid lines, and is represented symbolically here as the vector $\{X\}$. Consider a particular fluid-flow problem for which the computational mesh $\{X_1\}$ has been defined, and for which a conventional steady-state solution $\{Q_1^*\}$ is known; that is

$$\{R(Q_1^*, X_1)\} = \{0\} \tag{2}$$

Note in Eq. (2) that the explicit functional dependence of the steady-state discrete algebraic equations on the physical (x,y) coordinates of the mesh is now emphasized. Consider next a second fluid-flow problem, similar to the first except that the geometric shape is different. This defines a second mesh $\{X_2\}$ given by

$$\{R(Q_2^*, X_2)\} = \{0\} \tag{3}$$

A Taylor series expansion about $\{R(Q_1^*, X_1)\}$ is (neglecting higher-order terms)

$$\{R(Q_2^*, X_2)\} = \{R(Q_1^*, X_1)\} + \left[\frac{\partial R(Q_1^*, X_1)}{\partial Q}\right]\{\Delta Q^*\} + \left[\frac{\partial R(Q_1^*, X_1)}{\partial X}\right]\{\Delta X\} \tag{4}$$

where

$$\{\Delta Q^*\} = \{Q_2^*\} - \{Q_1^*\} \quad \text{and} \quad \{\Delta X\} = \{X_2\} - \{X_1\} \tag{5}$$

Since the steady-state solutions satisfy Eqs. (2) and (3), Eq. (4) becomes

$$-\left[\frac{\partial R(Q_1^*, X_1)}{\partial Q}\right]\{\Delta Q^*\} = \left[\frac{\partial R(Q_1^*, X_1)}{\partial X}\right]\{\Delta X\} \tag{6}$$

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